Quantum Gates Catch-up Problems

1. The Hadamard gate takes the form
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{1}$$
Show that $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$ and $X = HZH$.

2. a) Write out the truth tables and explicit matrices for the NOT and CNOT gates.

   b) Show that C-NOT can be written as
$$\text{C-NOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$$
.

   c) The C-Phase gate implements a Z gate on the second qubit conditional on the first qubit being in state $|1\rangle$. Write out the truth table and matrix for the C-Phase gate. The C-Phase gate is sometimes said to be a symmetric gate- why do you think this is?

   d) Show that C-Phase $= (\mathbb{1} \otimes H)(\text{C-NOT})(\mathbb{1} \otimes H)$. (Don't use matrix multiplication).

3. a) The SWAP gate swaps the state of system 1 and system 2, i.e. $\text{SWAP}|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Write out the explicit matrix form for SWAP.

   b) Show that 3 CNOTs can be used to implement the SWAP gate. (Do try and figure this out at first without google!)

4. a) Write out the four Bell states. Show that it is possible to transform between them using single qubit gates.

   b) Sketch the circuits you would use to prepare each of the Bell states from the all zero state $|00\rangle$.

   c) What is the effect of applying the SWAP gate to a Bell state?

   d) Suppose you measure the first qubit of the $|\Psi^-\rangle$ Bell state in the computational basis. What is the probability of finding the first qubit to be in the state $|1\rangle$? Suppose *after* this measurement you now measure the second qubit- what's the probability of finding it to be in the state $|1\rangle$?

   e) Show that the Bell states form an orthonormal basis. Draw a circuit to perform the $|\Psi^-\rangle\langle\Psi^-|$ projective measurement. What is the probability of measuring the state $|+-\rangle$ to be in the state $|\Psi^-\rangle$?